

Homogenization of monotone parabolic problems with an arbitrary number of spatial and temporal scales

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Abstract

In this paper we prove a general homogenization result for monotone parabolic problems with an arbitrary number of microscopic scales in space as well as in time, where the scale functions are not necessarily powers of epsilon. The main tools for the homogenization procedure are multiscale convergence and very weak multiscale convergence, both adapted to evolution problems. At the end of the paper an example is given to concretize the use of the main result.

1. Introduction

The mathematical theory of nonlinear partial differential equations plays an important role in e.g. applied mathematics and physics. In this paper we present a homogenization result for the general monotone parabolic problem with multiple spatial and temporal scales

$$\begin{aligned} \partial_t u^\varepsilon(x, t) - \nabla \cdot a\left(\frac{x}{\hat{\varepsilon}_1}, \dots, \frac{x}{\hat{\varepsilon}_n}, \frac{t}{\check{\varepsilon}_1}, \dots, \frac{t}{\check{\varepsilon}_m}, \nabla u^\varepsilon(x, t)\right) &= f(x, t) \text{ in } \Omega_T, \\ u^\varepsilon(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u^\varepsilon(x, 0) &= u^0(x) \text{ in } \Omega, \end{aligned} \quad (1)$$

where $f \in L^2(\Omega_T)$ and $u^0 \in L^2(\Omega)$. Here Ω is an open bounded set in \mathbb{R}^N with smooth boundary and $\Omega_T = \Omega \times (0, T)$. We let $Y = (0, 1)^N$ and $S = (0, 1)$ and we assume that a is Y -periodic in the n first variables and S -periodic in the following m variables. Finally we let $\hat{\varepsilon}_k$ for $k = 1, \dots, n$ and $\check{\varepsilon}_j$ for $j = 1, \dots, m$ be scale functions depending on ε that tend to zero as ε does, where the scales are assumed to fulfil certain conditions of separatedness.

The homogenization of (1) means studying the asymptotic behavior of the corresponding sequence of solutions u^ε as ε tends to zero and finding the limit

problem

$$\begin{aligned}\partial_t u(x, t) - \nabla \cdot b(x, t, \nabla u) &= f(x, t) \text{ in } \Omega_T, \\ u(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= u^0(x) \text{ in } \Omega,\end{aligned}$$

which admits the function u , the limit of $\{u^\varepsilon\}$, as its unique solution. Here b is characterized by local problems, one for each microscopic spatial scale. For more informative texts on homogenization theory we suggest e.g. [1], [4] and [16].

The main tools to carry out the homogenization process for (1) are multi-scale convergence and very weak multiscale convergence in the evolution setting. Here very weak multiscale convergence, see e.g. [9] and [12], is the key to handling the difficulties that appear when rapid time oscillations are present. The nonlinearity of the problem is treated by applying the perturbed test functions method.

Homogenization results for linear parabolic equations with oscillations in one spatial scale and one temporal scale were studied by using asymptotic expansions in [3]. In [14] parabolic problems containing fast oscillations in space as well as in time were treated for the first time applying two-scale convergence methods. Parabolic homogenization problems have also been investigated in e.g. [8] and [10] for different choices of fixed scales. Linear parabolic problems with an arbitrary number of scales in both space and time were homogenized in [12]. Homogenization results for monotone, not necessarily linear, problems have been presented in e.g. [7], [19], [13], [23] and [24]. The case with one spatial microscale and an arbitrary number of temporal scales was treated by Persson in [20].

The paper is organized in the following way. In Section 2 we give some preparatory theory concerning multiscale and very weak multiscale convergence. In Section 3 we present the homogenization result for (1) and in the last section we look at a special case of (1) to illustrate the use of the presented result.

Notation 1 We let $F_\sharp(Y)$ be the space of all functions in $F_{loc}(\mathbb{R}^N)$ which are the periodic repetition of some function in $F(Y)$. We also let $Y_k = Y$ for $k = 1, \dots, n$, $Y^n = Y_1 \times \dots \times Y_n$ (the nN -dimensional open unit cell), $y^n = y_1, \dots, y_n$ (corresponding spatial multivariable), $dy^n = dy_1 \dots dy_n$, $S_j = S$ for $j = 1, \dots, m$, $S^m = S_1 \times \dots \times S_m$ (the m -dimensional open unit cell), $s^m = s_1, \dots, s_m$ (corresponding local temporal multivariable), $ds^m = ds_1 \dots ds_m$ and $\mathcal{Y}_{n,m} = Y^n \times S^m$, where we interpret $\mathcal{Y}_{0,m}$ as S^m . We let $\hat{\varepsilon}_k(\varepsilon)$, for $k = 1, \dots, n$, and $\check{\varepsilon}_j(\varepsilon)$, $j = 1, \dots, m$, be strictly positive functions such that $\hat{\varepsilon}_k(\varepsilon)$ and $\check{\varepsilon}_j(\varepsilon)$ go to zero when ε does. We also use the notations $\hat{\varepsilon}^n = \hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n$ and $\check{\varepsilon}^m = \check{\varepsilon}_1, \dots, \check{\varepsilon}_m$ and furthermore $\frac{x}{\hat{\varepsilon}^n}$ denotes $\frac{x}{\hat{\varepsilon}_1}, \dots, \frac{x}{\hat{\varepsilon}_n}$ and, similarly, by $\frac{t}{\check{\varepsilon}^m}$ we mean $\frac{t}{\check{\varepsilon}_1}, \dots, \frac{t}{\check{\varepsilon}_m}$.

2. Multiscale and very weak multiscale convergence

In [17] Nguetseng presented a new homogenization technique based on a certain type of convergence which has become known as two-scale convergence. This was extended in [2] to so-called multiscale convergence, which allows use of multiple scales and makes it possible to capture numerous types of spatial microscopic oscillations. Below we define evolution multiscale convergence i.e., the concept has been further developed to include temporal oscillations, see [12].

Definition 2 *A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$ -scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if*

$$\begin{aligned} & \int_{\Omega_T} u^\varepsilon(x, t) v\left(x, t, \frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}\right) dx dt \\ & \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x, t, y^n, s^m) v(x, t, y^n, s^m) dy^n ds^m dx dt \end{aligned}$$

for any $v \in L^2(\Omega_T; C_\#(\mathcal{Y}_{n,m}))$. We write

$$u^\varepsilon(x, t) \xrightarrow{n+1, m+1} u_0(x, t, y^n, s^m).$$

Next we define some concepts regarding relations between scale functions.

Definition 3 *We say that the scales in a list $\{\varepsilon_1, \dots, \varepsilon_n\}$ are separated if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

for $k = 1, \dots, n-1$ and that the scales are well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^l = 0$$

for $k = 1, \dots, n-1$.

Definition 4 *Let $\{\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ and $\{\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_m\}$ be lists of (well-)separated scales. Collect all elements from both lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is (well-)separated, the lists $\{\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ and $\{\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_m\}$ are said to be jointly (well-)separated.*

We give the two following theorems, which state a compactness result for $(n+1, m+1)$ -scale convergence and a characterization of multiscale limits for gradients, respectively.

Theorem 5 *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and suppose that the lists $\{\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ and $\{\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_m\}$ are jointly separated. Then there exists a u_0 in $L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that, up to a subsequence,*

$$u^\varepsilon(x, t) \xrightarrow{n+1, m+1} u_0(x, t, y^n, s^m).$$

Proof. See Theorem 2.66 in [21] or Theorem A.1. in [12]. ■

The space $W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ that appears in the theorem below is the space of all functions in $L^2(0, T; H_0^1(\Omega))$ such that the time derivative belongs to $L^2(0, T; H^{-1}(\Omega))$.

Theorem 6 *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ and suppose that the lists $\{\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ and $\{\check{\varepsilon}_1, \dots, \check{\varepsilon}_m\}$ are jointly well-separated. Then, up to a subsequence,*

$$\begin{aligned} u^\varepsilon(x, t) &\rightarrow u(x, t) \text{ in } L^2(\Omega_T), \\ u^\varepsilon(x, t) &\rightharpoonup u(x, t) \text{ in } L^2(0, T; H_0^1(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x, t) \xrightarrow{n+1, m+1} \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^m)$$

where $u \in W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1, m}; H_\#^1(Y_j)/\mathbb{R})$ for $j = 1, \dots, n$.

Proof. See Theorem 2.74 in [21] or Theorem 4 in [12]. ■

Multiscale convergence is very useful for homogenization of problems involving rapid oscillations on several micro levels. Unfortunately, we can only use this for sequences that are bounded in the L^2 -norm but when rapid time oscillations are present we encounter sequences that do not possess this boundedness. Multiscale convergence has a large class of test functions and the limit captures both the global trend and the microscopic oscillations. If we downsize this class so that it only captures the microscopic fluctuations then it becomes possible to apply it to certain sequences that do not have to be bounded in any Lebesgue space. This is the idea behind so-called very weak multiscale convergence. A first compactness result of very weak multiscale convergence type was given in [14], see also [19], [9] and [11].

Definition 7 *A sequence $\{w^\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1, m+1)$ -scale converge very weakly to $w_0 \in L^1(\Omega_T \times \mathcal{Y}_{n, m})$ if*

$$\begin{aligned} &\int_{\Omega_T} w^\varepsilon(x, t) v_1\left(x, \frac{x}{\hat{\varepsilon}_1}, \dots, \frac{x}{\hat{\varepsilon}_{n-1}}\right) v_2\left(t, \frac{t}{\check{\varepsilon}_1}, \dots, \frac{t}{\check{\varepsilon}_m}\right) c\left(\frac{x}{\hat{\varepsilon}_n}\right) dx dt \\ &\rightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n, m}} w_0(x, t, y^n, s^m) v_1(x, y^{n-1}) v_2(t, s^m) c(y_n) dy^n ds^m dx dt \end{aligned}$$

for any $v_1 \in D(\Omega; C_\#^\infty(Y^{n-1}))$, $v_2 \in C_\#^\infty(Y_n)/\mathbb{R}$ and $c \in D(0, T; C_\#^\infty(S^m))$ where

$$\int_{Y_n} w_0(x, t, y^n, s^m) dy_n = 0.$$

We write

$$w^\varepsilon(x, t) \xrightarrow[n+1, m+1]{vw} w_0(x, t, y^n, s^m).$$

The following theorem is essential for the homogenization of (1).

Theorem 8 *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ and assume that the lists $\{\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$ and $\{\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_m\}$ are jointly well-separated. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x, t)}{\varepsilon_n} \xrightarrow[n+1, m+1]{vw} u_n(x, t, y^n, s^m),$$

where, for $n = 1, 2, \dots$, $u_n \in L^2(\Omega_T \times \mathcal{Y}_{n-1, m}; H_\#^1(Y_n)/\mathbb{R})$.

Proof. See Theorem 2.78 in [21] or Theorem 7 in [12]. ■

3. The homogenization result

We study the homogenization of the problem

$$\begin{aligned} \partial_t u^\varepsilon(x, t) - \nabla \cdot a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\tilde{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) &= f(x, t) \text{ in } \Omega_T, \\ u^\varepsilon(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u^\varepsilon(x, 0) &= u^0(x) \text{ in } \Omega, \end{aligned} \quad (2)$$

where $f \in L^2(\Omega_T)$ and $u^0 \in L^2(\Omega)$. Here we assume that

$$a : \mathbb{R}^{nN} \times \mathbb{R}^m \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

satisfies the following structure conditions, where C_0 and C_1 are positive constants and $0 < \alpha \leq 1$:

- (i) $a(y^n, s^m, 0) = 0$ for all $(y^n, s^m) \in \mathbb{R}^{nN} \times \mathbb{R}^m$.
- (ii) $a(\cdot, \cdot, \xi)$ is $\mathcal{Y}_{n, m}$ -periodic in (y^n, s^m) and continuous for all $\xi \in \mathbb{R}^N$.
- (iii) $a(y^n, s^m, \cdot)$ is continuous for all $(y^n, s^m) \in \mathbb{R}^{nN} \times \mathbb{R}^m$.
- (iv) $(a(y^n, s^m, \xi) - a(y^n, s^m, \xi')) \cdot (\xi - \xi') \geq C_0 |\xi - \xi'|^2$ for all $(y^n, s^m) \in \mathbb{R}^{nN} \times \mathbb{R}^m$ and all $\xi, \xi' \in \mathbb{R}^N$.
- (v) $|a(y^n, s^m, \xi) - a(y^n, s^m, \xi')| \leq C_1(1 + |\xi| + |\xi'|)^{1-\alpha} |\xi - \xi'|^\alpha$ for all $(y^n, s^m) \in \mathbb{R}^{nN} \times \mathbb{R}^m$ and all $\xi, \xi' \in \mathbb{R}^N$.

Finally we assume that the lists $\{\hat{\varepsilon}^n\}$ and $\{\tilde{\varepsilon}^m\}$ in (2) are jointly well-separated.

In order to formulate the theorem below in a neat way we define some numbers determined by how the scales functions present are related to each other. We define d_i and ρ_i , $i = 1, \dots, n$, as follows:

(I) If

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{\varepsilon}_1}{(\tilde{\varepsilon}_i)^2} = 0,$$

then $d_i = m$. If

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{\varepsilon}_j}{(\tilde{\varepsilon}_i)^2} > 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \frac{\tilde{\varepsilon}_{j+1}}{(\tilde{\varepsilon}_i)^2} = 0$$

for some $j = 1, \dots, m-1$, then $d_i = m-j$. If

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{\varepsilon}_m}{(\tilde{\varepsilon}_i)^2} > 0,$$

then $d_i = 0$.

(II) If

$$\lim_{\varepsilon \rightarrow 0} \frac{(\hat{\varepsilon}_i)^2}{\tilde{\varepsilon}_j} = C,$$

$0 < C < \infty$, for some $j = 1, \dots, m$ we say that we have resonance and we let $\rho_i = C$, otherwise $\rho_i = 0$.

This means that d_i is the number of temporal scales faster than the square of the spatial scale in question and ρ_i indicates whether there is resonance or not.

We are now prepared to give and prove the main theorem of the paper. Here $W_{2\sharp}^1(S; H_{\sharp}^1(Y)/\mathbb{R}, L_{\sharp}^2(Y)/\mathbb{R})$ denotes the space of all functions u such that $u \in L_{\sharp}^2(S; H_{\sharp}^1(Y)/\mathbb{R})$ and $\partial_s u \in L_{\sharp}^2(S; (H_{\sharp}^1(Y)/\mathbb{R})')$.

Theorem 9 *Let $\{u^\varepsilon\}$ be a sequence of solutions in $W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ to (2). Then it holds that*

$$u^\varepsilon(x, t) \rightarrow u(x, t) \text{ in } L^2(\Omega_T), \quad (3)$$

$$u^\varepsilon(x, t) \rightharpoonup u(x, t) \text{ in } L^2(0, T; H_0^1(\Omega)) \quad (4)$$

and

$$\nabla u^\varepsilon(x, t) \xrightarrow{n+1, m+1} \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}),$$

where $u \in W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ is the unique solution to

$$\begin{aligned} \partial_t u(x, t) - \nabla \cdot b(x, t, \nabla u) &= f(x, t) \text{ in } \Omega_T, \\ u(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= u^0(x) \text{ in } \Omega \end{aligned}$$

with

$$b(x, t, \nabla u) = \int_{\mathcal{Y}_{n,m}} a \left(y^n, s^m, \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \right) dy^n ds^m,$$

where $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1, m-d_i}; H_{\sharp}^1(Y_i)/\mathbb{R})$ for $i = 1, \dots, n$. Here u_i , for $i = 1, \dots, n$, are the unique solutions to the system of local problems

$$\begin{aligned} & \rho_i \partial_{s_{m-d_i}} u_i(x, t, y^i, s^{m-d_i}) \\ & - \nabla_{y_i} \cdot \int_{S_{m-d_i+1}} \cdots \int_{S_m} \int_{Y_{i+1}} \cdots \int_{Y_n} a \left(y^n, s^m, \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \right) \\ & \times dy_n \cdots dy_{i+1} ds_m \cdots ds_{m-d_i+1} = 0 \end{aligned} \quad (5)$$

if we assume that $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1, m-d_i-1}; W_{2_{\sharp}}^1(S_{m-d_i}; H_{\sharp}^1(Y_i)/\mathbb{R}, L_{\sharp}^2(Y_i)/\mathbb{R}))$ when $\rho_i \neq 0$.

Proof. The lists $\{\hat{\varepsilon}^n\}$ and $\{\tilde{\varepsilon}^m\}$ of scales are jointly well-separated and $\{u^\varepsilon\}$ is bounded in $W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$, see Proposition 3.16 in [21], which means that Theorem 6 is applicable and hence, up to a subsequence,

$$\begin{aligned} u^\varepsilon(x, t) & \rightarrow u(x, t) \text{ in } L^2(\Omega_T), \\ u^\varepsilon(x, t) & \rightharpoonup u(x, t) \text{ in } L^2(0, T; H_0^1(\Omega)) \end{aligned}$$

and

$$\nabla u^\varepsilon(x, t) \xrightarrow{n+1, m+1} \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^m),$$

where $u \in W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ and $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1, m}; H_{\sharp}^1(Y_j)/\mathbb{R})$ for $j = 1, \dots, n$.

The weak form of (2) reads: find $u^\varepsilon \in W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$ such that

$$\begin{aligned} & \int_{\Omega_T} -u^\varepsilon(x, t) v(x) \partial_t c(t) + a \left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\tilde{\varepsilon}^m}, \nabla u^\varepsilon(x, t) \right) \cdot \nabla v(x) c(t) dx dt \\ & = \int_{\Omega_T} f(x, t) v(x) c(t) dx dt \end{aligned} \quad (6)$$

for all $v \in H_0^1(\Omega)$ and $c \in D(0, T)$. By choosing $\xi' = 0$ in (v) we have

$$|a(y^n, s^m, \xi)| \leq C_1(1 + |\xi|)^{1-\alpha} |\xi|^\alpha$$

and since

$$C_1(1 + |\xi|)^{1-\alpha} |\xi|^\alpha < C_1(1 + |\xi|)^{1-\alpha} (1 + |\xi|)^\alpha$$

we obtain

$$|a(y^n, s^m, \xi)| \leq C_1(1 + |\xi|). \quad (7)$$

The boundedness of $\{u^\varepsilon\}$ in $L^2(0, T; H_0^1(\Omega))$ together with (7) gives, up to a subsequence, that

$$a \left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\tilde{\varepsilon}^m}, \nabla u^\varepsilon(x, t) \right) \xrightarrow{m, n} a_0(x, t, y^n, s^m)$$

for some $a_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ due to Theorem 5. We let ε tend to zero in (6) and obtain

$$\begin{aligned} \int_{\Omega_T} -u(x,t)v(x)\partial_t c(t) + \left(\int_{\mathcal{Y}_{n,m}} a_0(x,t,y^n,s^m) dy^n ds^m \right) \cdot \nabla v(x)c(t) \, dxdt \\ = \int_{\Omega_T} f(x,t)v(x)c(t) \, dxdt, \end{aligned} \quad (8)$$

which is the homogenized problem if we can prove that

$$a_0(x,t,y^n,s^m) = a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j \right)$$

with u and u_j as given in the theorem. To characterize a_0 we will use the system of local problems (5), and deriving this will be our next aim.

In (6) we will use test functions defined according to the following. Let $r_\varepsilon = r(\varepsilon)$ be a sequence of positive numbers tending to zero as ε does. Fix $i = 1, \dots, n$ and choose

$$v(x) = r_\varepsilon v_1(x) v_2 \left(\frac{x}{\hat{\varepsilon}_1} \right) \cdots v_{i+1} \left(\frac{x}{\hat{\varepsilon}_i} \right)$$

and

$$c(t) = c_1(t) c_2 \left(\frac{t}{\check{\varepsilon}_1} \right) \cdots c_{\lambda+1} \left(\frac{t}{\check{\varepsilon}_\lambda} \right), \quad \lambda = 1, \dots, m$$

with $v_1 \in D(\Omega)$, $v_j \in C_\#^\infty(Y_{j-1})$ for $j = 2, \dots, i$, $v_{i+1} \in C_\#^\infty(Y_i)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_l \in C_\#^\infty(S_{l-1})$ for $l = 2, \dots, \lambda+1$. We get

$$\begin{aligned} \int_{\Omega_T} -u^\varepsilon(x,t)v_1(x)v_2 \left(\frac{x}{\hat{\varepsilon}_1} \right) \cdots v_{i+1} \left(\frac{x}{\hat{\varepsilon}_i} \right) \\ \times \left(r_\varepsilon \partial_t c_1(t) c_2 \left(\frac{t}{\check{\varepsilon}_1} \right) \cdots c_{\lambda+1} \left(\frac{t}{\check{\varepsilon}_\lambda} \right) \right. \\ \left. + \sum_{l=2}^{\lambda+1} \frac{r_\varepsilon}{\check{\varepsilon}_{l-1}} c_1(t) c_2 \left(\frac{t}{\check{\varepsilon}_1} \right) \cdots \partial_{s_{l-1}} c_l \left(\frac{t}{\check{\varepsilon}_{l-1}} \right) \cdots c_{\lambda+1} \left(\frac{t}{\check{\varepsilon}_\lambda} \right) \right) \\ + a \left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x,t) \right) \cdot \left(r_\varepsilon \nabla v_1(x) v_2 \left(\frac{x}{\hat{\varepsilon}_1} \right) \cdots v_{i+1} \left(\frac{x}{\hat{\varepsilon}_i} \right) \right. \\ \left. + \sum_{j=2}^{i+1} \frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}} v_1(x) v_2 \left(\frac{x}{\hat{\varepsilon}_1} \right) \cdots \nabla_{y_{j-1}} v_j \left(\frac{x}{\hat{\varepsilon}_{j-1}} \right) \cdots v_{i+1} \left(\frac{x}{\hat{\varepsilon}_i} \right) \right) \\ \times c_1(t) c_2 \left(\frac{t}{\check{\varepsilon}_1} \right) \cdots c_{\lambda+1} \left(\frac{t}{\check{\varepsilon}_\lambda} \right) dxdt \\ = \int_{\Omega_T} f(x,t) r_\varepsilon v_1(x) v_2 \left(\frac{x}{\hat{\varepsilon}_1} \right) \cdots v_{i+1} \left(\frac{x}{\hat{\varepsilon}_i} \right) \\ \times c_1(t) c_2 \left(\frac{t}{\check{\varepsilon}_1} \right) \cdots c_{\lambda+1} \left(\frac{t}{\check{\varepsilon}_\lambda} \right) dxdt. \end{aligned}$$

Applying Theorem 6 and the definition of r_ε , we may let $\varepsilon \rightarrow 0$ and get

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -u^\varepsilon(x, t) v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times \left(\sum_{l=2}^{\lambda+1} \frac{r_\varepsilon}{\check{\varepsilon}_{l-1}} c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\check{\varepsilon}_{l-1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) \right) \\
& + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\
& \cdot \sum_{j=2}^{i+1} \frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots \nabla_{y_{j-1}} v_j\left(\frac{x}{\hat{\varepsilon}_{j-1}}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) dx dt = 0
\end{aligned}$$

if we omit the terms passing to zero. Rewriting we obtain

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x, t) \sum_{l=2}^{\lambda+1} \frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_{l-1}} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_{l-1}} c_l\left(\frac{t}{\check{\varepsilon}_{l-1}}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) \\
& + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\
& \cdot \sum_{j=2}^{i+1} \frac{r_\varepsilon}{\hat{\varepsilon}_{j-1}} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots \nabla_{y_{j-1}} v_j\left(\frac{x}{\hat{\varepsilon}_{j-1}}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) dx dt = 0,
\end{aligned} \tag{9}$$

where we have factored out $\frac{1}{\hat{\varepsilon}_i}$ from the first sum to make it obvious that it is possible to pass to the limit by means of very weak $(i+1, \lambda+1)$ -scale convergence. Suppose that $\{\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda}\}$ and $\{\frac{r_\varepsilon}{\hat{\varepsilon}_i}\}$ are bounded. This implies that

$$\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_{\lambda-j}} \rightarrow 0, \quad j = 1, \dots, \lambda - 1$$

and

$$\frac{r_\varepsilon}{\hat{\varepsilon}_{i-j}} \rightarrow 0, \quad j = 1, \dots, i - 1$$

as $\varepsilon \rightarrow 0$ due to the fact that the scales are separated. Hence, under these

assumptions (9) turns into

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x, t) \frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_\lambda} c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\
& \cdot \frac{r_\varepsilon}{\hat{\varepsilon}_i} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_i\left(\frac{x}{\hat{\varepsilon}_{i-1}}\right) \nabla_{y_i} v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) dx dt = 0,
\end{aligned} \tag{10}$$

which will be our springboard when deriving both the independencies of the local time variables in the corrector functions and the local problems. This will be done for the two different cases nonresonance and resonance.

Case 1: Nonresonance ($\rho_i = 0$). First we derive the independencies for $d_i > 0$. Let λ successively be $m, \dots, m - d_i + 1$. If $r_\varepsilon = \frac{\check{\varepsilon}_\lambda}{\hat{\varepsilon}_i}$ we have from the chosen values of λ and the meaning of d_i that

$$\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda} = 1$$

and

$$\frac{r_\varepsilon}{\hat{\varepsilon}_i} = \frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2} \rightarrow 0 \tag{11}$$

as $\varepsilon \rightarrow 0$. Hence, we may use (10) for this choice of r_ε and we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x, t) v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_\lambda} c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\
& \cdot \frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_i\left(\frac{x}{\hat{\varepsilon}_{i-1}}\right) \nabla_{y_i} v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\
& \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) dx dt = 0.
\end{aligned}$$

We let ε tend to zero and obtain, due to Theorem 8 and (11), that

$$\begin{aligned}
& \int_{\Omega_T} \int_{\mathcal{Y}_{i,\lambda}} -u_i(x, t, y^i, s^\lambda) v_1(x) v_2(y_1) \cdots v_{i+1}(y_i) \\
& \times c_1(t) c_2(s_1) \cdots \partial_{s_\lambda} c_{\lambda+1}(s_\lambda) dy^i ds^\lambda dx dt = 0
\end{aligned}$$

and by the variational lemma we have

$$\int_{S_\lambda} -u_i(x, t, y^i, s^\lambda) \partial_{s_\lambda} c_{\lambda+1}(s_\lambda) ds_\lambda = 0$$

almost everywhere for all $c_{\lambda+1} \in C_{\#}^{\infty}(s_{\lambda})$. This means that u_i is independent of s_{m-d_i+1}, \dots, s_m .

We proceed by deriving the local problems and for this purpose we choose $r_{\varepsilon} = \hat{\varepsilon}_i$ and $\lambda = m - d_i$, where $d_i \geq 0$. Since $d_i \geq 0$ and $\rho_i = 0$ we conclude that

$$\frac{r_{\varepsilon} \hat{\varepsilon}_i}{\check{\varepsilon}_{\lambda}} = \frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$ and

$$\frac{r_{\varepsilon}}{\hat{\varepsilon}_i} = 1,$$

which means that (10) is valid and we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^{\varepsilon}(x, t) \frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_{m-d_i}} c_{m-d_i+1}\left(\frac{t}{\check{\varepsilon}_{m-d_i}}\right) + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^{\varepsilon}(x, t)\right) \\ & \cdot v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_i\left(\frac{x}{\hat{\varepsilon}_{i-1}}\right) \nabla_{y_i} v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{m-d_i+1}\left(\frac{t}{\check{\varepsilon}_{m-d_i}}\right) dx dt = 0. \end{aligned}$$

As $\varepsilon \rightarrow 0$ we obtain

$$\begin{aligned} & \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} a_0(x, t, y^n, s^m) \\ & \cdot v_1(x) v_2(y_1) \cdots v_i(y_{i-1}) \nabla_{y_i} v_{i+1}(y_i) \\ & \times c_1(t) c_2(s_1) \cdots c_{m-d_i+1}(s_{m-d_i}) dy^n ds^m dx dt = 0 \end{aligned}$$

and, finally,

$$\begin{aligned} & \int_{S_{m-d_i+1}} \cdots \int_{S_m} \int_{Y_i} \cdots \int_{Y_n} a_0(x, t, y^n, s^m) \\ & \cdot \nabla_{y_i} v_{i+1}(y_i) dy_n \cdots dy_i ds_m \cdots ds_{m-d_i+1} = 0 \end{aligned} \tag{12}$$

almost everywhere for all $v_{i+1} \in H_{\#}^1(Y_i)/\mathbb{R}$, which is the weak form of the local problem in this nonresonance case.

Case 2: Resonance ($\rho_i = C$). As in the first case we begin with the independencies for $d_i > 0$. Again, let λ successively be $m, \dots, m - d_i + 1$. Now choose $r_{\varepsilon} = \frac{\check{\varepsilon}_{\lambda}}{\hat{\varepsilon}_i}$ directly implying that

$$\frac{r_{\varepsilon} \hat{\varepsilon}_i}{\check{\varepsilon}_{\lambda}} = 1$$

and

$$\frac{r_{\varepsilon}}{\hat{\varepsilon}_i} = \frac{\check{\varepsilon}_{\lambda}}{(\hat{\varepsilon}_i)^2} \rightarrow 0$$

when $\varepsilon \rightarrow 0$, by the restriction of λ and the definition of d_i and ρ_i . Thus, (10) turns into

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x, t) v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_\lambda} c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\ & \cdot \frac{\check{\varepsilon}_\lambda}{(\hat{\varepsilon}_i)^2} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_i\left(\frac{x}{\hat{\varepsilon}_{i-1}}\right) \nabla_{y_i} v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{\lambda+1}\left(\frac{t}{\check{\varepsilon}_\lambda}\right) dx dt = 0 \end{aligned}$$

and a passage to the limit gives

$$\begin{aligned} & \int_{\Omega_T} \int_{\mathcal{Y}_{i,\lambda}} -u_i(x, t, y^i, s^\lambda) v_1(x) v_2(y_1) \cdots v_{i+1}(y_i) \\ & \times c_1(t) c_2(s_1) \cdots \partial_{s_\lambda} c_{\lambda+1}(s_\lambda) dy^i ds^\lambda dx dt = 0. \end{aligned}$$

Hence,

$$\int_{S_\lambda} -u_i(x, t, y^i, s^\lambda) \partial_{s_\lambda} c_{\lambda+1}(s_\lambda) ds_\lambda = 0$$

almost everywhere for all $c_{\lambda+1} \in C_\#^\infty(S_\lambda)$, and thus u_i is independent of s_λ .

To extract the local problem we choose $r_\varepsilon = \hat{\varepsilon}_i$ and $\lambda = m - d_i$, where $d_i \geq 0$, which gives

$$\frac{r_\varepsilon \hat{\varepsilon}_i}{\check{\varepsilon}_\lambda} = \frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}} \rightarrow \rho_i$$

as $\varepsilon \rightarrow 0$ and

$$\frac{r_\varepsilon}{\hat{\varepsilon}_i} = 1$$

and from (10) we then have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega_T} -\frac{1}{\hat{\varepsilon}_i} u^\varepsilon(x, t) \frac{(\hat{\varepsilon}_i)^2}{\check{\varepsilon}_{m-d_i}} v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots \partial_{s_{m-d_i}} c_{m-d_i+1}\left(\frac{t}{\check{\varepsilon}_{m-d_i}}\right) + a\left(\frac{x}{\hat{\varepsilon}^n}, \frac{t}{\check{\varepsilon}^m}, \nabla u^\varepsilon(x, t)\right) \\ & \cdot v_1(x) v_2\left(\frac{x}{\hat{\varepsilon}_1}\right) \cdots v_i\left(\frac{x}{\hat{\varepsilon}_{i-1}}\right) \nabla_{y_i} v_{i+1}\left(\frac{x}{\hat{\varepsilon}_i}\right) \\ & \times c_1(t) c_2\left(\frac{t}{\check{\varepsilon}_1}\right) \cdots c_{m-d_i+1}\left(\frac{t}{\check{\varepsilon}_{m-d_i}}\right) dx dt = 0. \end{aligned}$$

Letting ε tend to zero and applying Theorem 8 we obtain

$$\begin{aligned} & \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} -\rho_i u_i(x, t, y^i, s^{m-d_i}) v_1(x) v_2(y_1) \cdots v_{i+1}(y_i) \\ & \times c_1(t) c_2(s_1) \cdots \partial_{s_{m-d_i}} c_{m-d_i+1}(s_{m-d_i}) + a_0(x, t, y^n, s^m) \\ & \cdot v_1(x) v_2(y_1) \cdots v_i(y_{i-1}) \nabla_{y_i} v_{i+1}(y_i) \\ & \times c_1(t) c_2(s_1) \cdots c_{m-d_i+1}(s_{m-d_i}) dy^n ds^m dx dt = 0 \end{aligned}$$

and hence, we end up with

$$\begin{aligned} & \int_{S_{m-d_i}} \cdots \int_{S_m} \int_{Y_i} \cdots \int_{Y_n} -\rho_i u_i(x, t, y^i, s^{m-d_i}) v_{i+1}(y_i) \\ & \times \partial_{s_{m-d_i}} c_{m-d_i+1}(s_{m-d_i}) + a_0(x, t, y^n, s^m) \\ & \cdot \nabla_{y_i} v_{i+1}(y_i) c_{m-d_i+1}(s_{m-d_i}) dy_n \cdots dy_i ds_m \cdots ds_{m-d_i} = 0 \end{aligned} \quad (13)$$

almost everywhere for all $v_{i+1} \in H_{\#}^1(Y_i)/\mathbb{R}$ and $c_{m-d_i+1} \in C_{\#}^{\infty}(S_{m-d_i})$, the weak form of the local problem in this second case.

What remains is to characterize a_0 and to this end we use perturbed test functions, see [5] and [6], according to

$$\begin{aligned} & p^k(x, t, y^j, s^m) \\ & = p^{k,0}(x, t) + \sum_{j=1}^n p^{k,j}(x, t, y^j, s^{m-d_j}) + \delta c(x, t, y^n, s^m), \end{aligned}$$

where $p^{k,0} \in D(\Omega_T)^N$, $p^{k,j} \in D(\Omega_T; C_{\#}^{\infty}(\mathcal{Y}_{j,m-d_j}^N))$ for $j = 1, \dots, n$, $c \in D(\Omega_T; C_{\#}^{\infty}(\mathcal{Y}_{n,m}))^N$ and $\delta > 0$. We choose these sequences such that

$$\begin{aligned} & p^{k,0}(x, t) \rightarrow \nabla u(x, t) \text{ in } L^2(\Omega_T)^N, \\ & p^{k,j}(x, t, y^j, s^{m-d_j}) \rightarrow \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \text{ in } L^2(\Omega_T \times \mathcal{Y}_{j,m-d_j})^N \end{aligned}$$

and such that they converge almost everywhere to the same limits as $k \rightarrow \infty$, see p. 388 in [15]. We introduce the notation

$$p_{\varepsilon}^k(x, t) = p^k\left(x, t, \frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}\right).$$

Using property (iv) we get

$$\left(a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, \nabla u^{\varepsilon} \right) - a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, p_{\varepsilon}^k \right) \right) \cdot (\nabla u^{\varepsilon}(x, t) - p_{\varepsilon}^k(x, t)) \geq 0$$

and integration and expansion leads to

$$\begin{aligned} & \int_{\Omega_T} a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, \nabla u^{\varepsilon} \right) \cdot \nabla u^{\varepsilon}(x, t) - a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, \nabla u^{\varepsilon} \right) \cdot p_{\varepsilon}^k(x, t) \\ & - a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, p_{\varepsilon}^k \right) \cdot \nabla u^{\varepsilon}(x, t) \\ & + a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, p_{\varepsilon}^k \right) \cdot p_{\varepsilon}^k(x, t) dx dt \geq 0. \end{aligned} \quad (14)$$

Due to Theorem 30.A (c) in [22] we may replace vc with u^ε in (6) and get another way of expressing the first term in (14) and hence it can be written as

$$\begin{aligned} & \int_{\Omega_T} f(x, t) u^\varepsilon(x, t) - a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, \nabla u^\varepsilon\right) \cdot p_\varepsilon^k(x, t) \\ & - a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, p_\varepsilon^k\right) \cdot \nabla u^\varepsilon(x, t) + a\left(\frac{x}{\varepsilon^n}, \frac{t}{\varepsilon^m}, p_\varepsilon^k\right) \cdot p_\varepsilon^k(x, t) \, dx dt \\ & - \int_0^T \langle \partial_t u^\varepsilon(t), u^\varepsilon(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt \geq 0. \end{aligned} \quad (15)$$

We note that p^k , $a(y^n, s^m, p^k)$ and their product are admissible test functions and since

$$- \liminf_{\varepsilon \rightarrow 0} \int_0^T \langle \partial_t u^\varepsilon(t), u^\varepsilon(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt \leq - \int_0^T \langle \partial_t u(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt$$

(see p. 12–13 in [18]) we get, up to a subsequence, that

$$\begin{aligned} & \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} f(x, t) u(x, t) - a_0(x, t, y^n, s^m) \cdot p^k(x, t, y^n, s^m) \\ & - a(y^n, s^m, p^k) \cdot \left(\nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \right) \\ & + a(y^n, s^m, p^k) \cdot p^k(x, t, y^n, s^m) \, dy^n ds^m dx dt \\ & - \int_0^T \langle \partial_t u(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt \geq 0 \end{aligned} \quad (16)$$

when ε tends to zero. We proceed by letting k tend to infinity. From the choice of p^k we have that

$$p^k(x, t, y^n, s^m) \rightarrow \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) + \delta c(x, t, y^n, s^m)$$

in $L^2(\Omega_T \times \mathcal{Y}_{n,m})^N$ and almost everywhere in $\Omega_T \times \mathcal{Y}_{n,m}$. Furthermore

$$a(y^n, s^m, p^k) \rightarrow a\left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c\right)$$

almost everywhere in $\Omega_T \times \mathcal{Y}_{n,m}$ and hence

$$\begin{aligned} & a(y^n, s^m, p^k) \cdot p^k(x, t, y^n, s^m) \rightarrow a\left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c\right) \\ & \cdot \left(\nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \right) + \delta c(x, t, y^n, s^m) \end{aligned}$$

almost everywhere in $\Omega_T \times \mathcal{Y}_{n,m}$. When we pass to the limit in (16) we will use Lebesgue's generalized majorized convergence theorem for the third and fourth term where we go through the details for the fourth term. Choosing $\xi = p^k$ in (7) we have that

$$|a(y^n, s^m, p^k)| \leq C_1(1 + |p^k(x, t, y^n, s^m)|). \quad (17)$$

Successively applying Cauchy-Schwarz inequality and (17) we get

$$\begin{aligned} |a(y^n, s^m, p^k) \cdot p^k(x, t, y^n, s^m)| &\leq |a(y^n, s^m, p^k)| |p^k(x, t, y^n, s^m)| \\ &\leq C_1(1 + |p^k(x, t, y^n, s^m)|) |p^k(x, t, y^n, s^m)| \\ &= C_1 \left(|p^k(x, t, y^n, s^m)| + |p^k(x, t, y^n, s^m)|^2 \right). \end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$\begin{aligned} &\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} |p^k(x, t, y^n, s^m)| + |p^k(x, t, y^n, s^m)|^2 dy^n ds^m dx dt \\ &\rightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \left| \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) + \delta c(x, t, y^n, s^m) \right| \\ &+ \left| \nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) + \delta c(x, t, y^n, s^m) \right|^2 dy^n ds^m dx dt \end{aligned}$$

and hence, by Lebesgue's generalized majorized convergence theorem we conclude that

$$\begin{aligned} &\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} a(y^n, s^m, p^k) \cdot p^k(x, t, y^n, s^m) dy^n ds^m dx dt \\ &\rightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c \right) \\ &\cdot \left(\nabla u(x, t) + \sum_{j=1}^n \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) + \delta c(x, t, y^n, s^m) \right) dy^n ds^m dx dt. \end{aligned}$$

Thus, as k tends to infinity in (16) we find that

$$\begin{aligned}
& \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} f(x,t)u(x,t) - a_0(x,t,y^n,s^m) \\
& \cdot \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta c(x,t,y^n,s^m) \right) \\
& - a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c \right) \\
& \cdot \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) \right) + a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c \right) \\
& \cdot \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta c(x,t,y^n,s^m) \right) dy^n ds^m dx dt \\
& - \int_0^T \langle \partial_t u(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt \geq 0,
\end{aligned}$$

where some terms vanishes directly and we have

$$\begin{aligned}
& \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} f(x,t)u(x,t) - a_0(x,t,y^n,s^m) \\
& \cdot \left(\nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j}) + \delta c(x,t,y^n,s^m) \right) \quad (18) \\
& + a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c \right) \cdot \delta c(x,t,y^n,s^m) dy^n ds^m dx dt \\
& - \int_0^T \langle \partial_t u(t), u(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt \geq 0.
\end{aligned}$$

If we replace vc by u in (8) we get

$$\begin{aligned}
& \int_0^T \langle \partial_t u, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt + \int_{\Omega_T} \left(\int_{\mathcal{Y}_{n,m}} a_0(x,t,y^n,s^m) dy^n ds^m \right) \cdot \nabla u(x,t) dx dt \\
& = \int_{\Omega_T} f(x,t)u(x,t) dx dt \quad (19)
\end{aligned}$$

and with (19) in (18) we obtain

$$\begin{aligned}
& \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \sum_{j=1}^n -a_0(x, t, y^n, s^m) \cdot \nabla_{y_j} u_j(x, t, y^j, s^{m-d_j}) \\
& \quad - a_0(x, t, y^n, s^m) \cdot \delta c(x, t, y^n, s^m) \\
& + a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j \right) \cdot \delta c(x, t, y^n, s^m) dy^n ds^m dx dt \geq 0.
\end{aligned} \tag{20}$$

Using the local problems we will eliminate the first n terms in (20). We study them one at the time by letting j successively be equal to $1, \dots, n$. If $\rho_j = 0$ we use the local problem (12) from case 1 with $i = j$ and the corresponding term vanishes directly. If $\rho_j \neq 0$ then, by assumption, $u_j \in L^2(\Omega_T \times \mathcal{Y}_{j-1, m-d_j-1}, W_{2\sharp}^1(S_{m-d_j}; H_{\sharp}^1(Y_j)/\mathbb{R}, L_{\sharp}^2(Y_j)/\mathbb{R}))$, which implies that $u_j(x, t, y^{j-1}) \in W_{2\sharp}^1(S_{m-d_j}; H_{\sharp}^1(Y_j)/\mathbb{R}, L_{\sharp}^2(Y_j)/\mathbb{R})$. Then, from (13) with $i = j$, we obtain that

$$\begin{aligned}
& \rho_j \partial_{s_{m-d_j}} u_j(x, t, y^j, s^{m-d_j}) \\
& = \nabla_{y_j} \cdot \left(\int_{S_{m-d_j+1}} \dots \int_{S_m} \int_{Y_{j+1}} \dots \int_{Y_n} a_0(x, t, y^n, s^m) dy_n \dots dy_{j+1} ds_m \dots ds_{m-d_j+1} \right)
\end{aligned}$$

i.e. $\int_{S_{m-d_j+1}} \dots \int_{S_m} \int_{Y_{j+1}} \dots \int_{Y_n} -a_0(x, t, y^n, s^m) dy_n \dots dy_{j+1} ds_m \dots ds_{m-d_j+1} \nabla_{y_j}$ in (20) can be replaced with the derivative $\rho_j \partial_{s_{m-d_j}} u_j$. Thus, Corollary 4.1 in [19] yields that the term in question vanishes. What remains of (20) is

$$\begin{aligned}
& \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} \left(-a_0(x, t, y^n, s^m) + a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j + \delta c \right) \right) \\
& \quad \cdot \delta c(x, t, y^n, s^m) dy^n ds^m dx dt \geq 0.
\end{aligned}$$

Dividing by δ and passing to the limit in the sense of letting δ tend to zero, we deduce that

$$a_0(x, t, y^n, s^m) = a \left(y^n, s^m, \nabla u + \sum_{j=1}^n \nabla_{y_j} u_j \right).$$

Finally, by the uniqueness of u , the whole sequence converges and the proof is complete. ■

4. An illustrative example

In this section we investigate a specific nonlinear parabolic problem with a number of rapid spatial and temporal scales, some of which are not powers of ε . More precisely we consider the (3,4)-scaled problem

$$\begin{aligned} \partial_t u^\varepsilon(x, t) - \nabla \cdot a\left(\frac{x}{2\sqrt{\varepsilon}}, \frac{x}{\varepsilon^2}, \frac{t}{e^\varepsilon - 1}, \frac{t}{\ln(1 + \varepsilon^2)}, \frac{t}{\varepsilon^3 \ln(1 + \frac{1}{\varepsilon})}, \nabla u^\varepsilon(x, t)\right) &= f(x, t) \text{ in } \Omega_T, \\ u^\varepsilon(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u^\varepsilon(x, 0) &= u^0(x) \text{ in } \Omega. \end{aligned}$$

To apply Theorem 9 we must be reassured that the two lists $\{2\sqrt{\varepsilon}, \varepsilon^2\}$ and $\{e^\varepsilon - 1, \ln(1 + \varepsilon^2), \varepsilon^3 \ln(1 + \frac{1}{\varepsilon})\}$ are jointly well-separated. It holds that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{2\sqrt{\varepsilon}} \left(\frac{\varepsilon^2}{2\sqrt{\varepsilon}} \right)^1 &= 0, \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{e^\varepsilon - 1} \left(\frac{\ln(1 + \varepsilon^2)}{e^\varepsilon - 1} \right)^3 &= 0 \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\ln(1 + \varepsilon^2)} \left(\frac{\varepsilon^3 \ln(1 + \frac{1}{\varepsilon})}{\ln(1 + \varepsilon^2)} \right)^3 = 0,$$

which implies that both the spatial and temporal scales are well-separated. Moreover,

$$\lim_{\varepsilon \rightarrow 0} \frac{\ln(1 + \varepsilon^2)}{\varepsilon^2} = 1,$$

so we can remove duplicates and make the joint list $\{2\sqrt{\varepsilon}, e^\varepsilon - 1, \varepsilon^2, \varepsilon^3 \ln(1 + \frac{1}{\varepsilon})\}$, which is well-separated. According to Definition 4 this shows that our lists of scales are jointly well-separated. For the rest we assume that our problem fulfils the assumptions of Theorem 9.

To begin with, from Theorem 9 we know that the convergence results (3) and (4) hold, i.e. that

$$u^\varepsilon(x, t) \rightarrow u(x, t) \text{ in } L^2(\Omega_T)$$

and

$$u^\varepsilon(x, t) \rightharpoonup u(x, t) \text{ in } L^2(0, T; H_0^1(\Omega)).$$

To determine the independencies and make the local problems more precise, we need to identify which values of d_i and ρ_i to use. We recall that d_i is the number of temporal scales faster than the square of the spatial scale in question and ρ_i indicates whether there is resonance or not. Let us start with the slowest spatial scale, i.e. $i = 1$. To find d_1 we investigate on the basis of (I) how the first spatial scale is related to the temporal scales present in the problem. We have

$$\lim_{\varepsilon \rightarrow 0} \frac{e^\varepsilon - 1}{(2\sqrt{\varepsilon})^2} = \frac{1}{4} > 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{\ln(1 + \varepsilon^2)}{(2\sqrt{\varepsilon})^2} = 0,$$

which means that $d_1 = 2$. For the scale in question we have resonance since

$$\lim_{\varepsilon \rightarrow 0} \frac{(2\sqrt{\varepsilon})^2}{e^\varepsilon - 1} = 4,$$

i.e., $\rho_1 = 4$ according to (II). For $i = 2$ we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^3 \ln(1 + \frac{1}{\varepsilon})}{(\varepsilon^2)^2} = \infty$$

and hence $d_2 = 0$. We also observe that

$$\lim_{\varepsilon \rightarrow 0} \frac{(\varepsilon^2)^2}{\varepsilon^3 \ln(1 + \frac{1}{\varepsilon})} = 0,$$

which means that $\rho_2 = 0$.

Now from Theorem 9 we have

$$\nabla u^\varepsilon(x, t) \stackrel{3,4}{\rightharpoonup} \nabla u(x, t) + \nabla_{y_1} u_1(x, t, y_1, s_1) + \nabla_{y_2} u_2(x, t, y^2, s^3),$$

where $u \in W_2^1(0, T; H_0^1(\Omega), L^2(\Omega))$, $u_1 \in L^2(\Omega_T; W_{2\#}^1(S_1; H_\#^1(Y_1)/\mathbb{R}, L_\#^2(Y_1)/\mathbb{R}))$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,3}; H_\#^1(Y_2)/\mathbb{R})$. Here u is the unique solution to

$$\begin{aligned} \partial_t u(x, t) - \nabla \cdot b(x, t, \nabla u) &= f(x, t) \text{ in } \Omega_T, \\ u(x, t) &= 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= u^0(x) \text{ in } \Omega \end{aligned}$$

with

$$\begin{aligned} &b(x, t, \nabla u) \\ &= \int_{Y_{2,3}} a(y^2, s^3, \nabla u(x, t) + \nabla_{y_1} u_1(x, t, y_1, s_1) + \nabla_{y_2} u_2(x, t, y^2, s^3)) dy^2 ds^3 \end{aligned}$$

and we have the two local problems

$$\begin{aligned} &4\partial_{s_1} u_1(x, t, y_1, s_1) - \nabla_{y_1} \cdot \int_{S_2} \int_{S_3} \int_{Y_2} a(y^2, s^3, \nabla u(x, t) \\ &+ \nabla_{y_1} u_1(x, t, y_1, s_1) + \nabla_{y_2} u_2(x, t, y^2, s^3)) dy_2 ds_3 ds_2 = 0 \end{aligned}$$

and

$$-\nabla_{y_2} \cdot a(y^2, s^3, \nabla u(x, t) + \nabla_{y_1} u_1(x, t, y_1, s_1) + \nabla_{y_2} u_2(x, t, y^2, s^3)) = 0.$$

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